The Maslov correction in the semiclassical Feynman integral

P. A. Horváthy*
Laboratoire de Mathématiques et de Physique Théorique
Université de Tours
Parc de Grandmont
F-37 200 TOURS (France)

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Abstract

The Maslov correction to the wave function is the jump of $-\pi/2$ in the phase when the system passes through a caustic point. This phenomenon is related to the second variation and to the geometry of paths, as conveniently explained in Feynman's path integral framework. The results can be extended to any system using the semiclassical approximation. The 1-dimensional harmonic oscillator is used to illustrate the different derivations reviewed here.

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1 Introduction

A fascinating fact, first observed at the end of the 19th century [1], is that, after passing through a focal point, the phase of light jumps by $-\pi/2$. If a light beam is split into two parts one of them passes through a focal point while the other does not, then, when the two partial waves are recombined, a destructive interference is observed.

This curious phenomenon has later been extended to the quantum mechanics of massive particles, where it is referred to as the "Maslov correction" [2, 3, 4]. A simple illustration is provided by the harmonic oscillator [5]. Feynman's path integral framework [6] is ideally suited to understand how this comes about [7].

The aim of this Review is to derive the Maslov correction and study some if its aspects in the path integral framework.

2 The Feynman propagator of the oscillator

Let us first consider, for simplicity, a problem in 1 space dimension. Let x_1 and x_2 two points and let T > 0 a time interval ¹. Let us indeed consider the set, denoted by \mathcal{P} of all curves $\gamma(t)$ between x_1 and x_2 , i. e. such that $\gamma(0) = x_1$ and $\gamma(T) = x_2$. The classical action is a real valued function defined on \mathcal{P} ,

$$S(\gamma) = \int_0^T L(\gamma(t), \dot{\gamma}(t))dt, \qquad (2.1)$$

^{*}e-mail: horvathv@lmpt.univ-tours.fr

¹It is tacitly assumed that the system is conservative. Were this not the case, one should consider two instants t_1 and $t_2 > t_1$ instead of $T = t_2 - t_1$.

where $L(x, \dot{x})$ is the Lagrange function of the system. Then the Principle of Least Action (Hamilton's Principle) tells us that the actual motion which starts in x_1 and arrives to x_2 in T is the $\bar{\gamma}(t)$ that makes S stationary.

Apart of exceptional cases, (see below) the two given points x_1 to x_2 are joined, in time T, by a unique classical motion curve $\bar{\gamma}(t)$. But in Quantum Mechanics the situation is different. Intuitively, a quantum particle moves not along a unique path, but along *all paths* which join these points in the given time! Feynman, in his Thesis, argued indeed that to any path γ in \mathcal{P} is associated a complex number, namely

$$\exp\left[\frac{i}{\hbar}S(\gamma)\right],\tag{2.2}$$

where S denotes the classical action (2.1) calculated along γ . Next, if the "amplitude" (whose square $|\psi|^2$ is the probability) of finding our particle in the instant t=0 at the point x_1 is $\psi(x_1)$, then the amplitude, $\psi_T(x_2)$, of finding it at x_2 at time T will be

$$U_T \psi(x_2) \equiv \psi_T(x_2) = \int_{-\infty}^{\infty} K(x_2, T | x_1, 0) \psi(x_1) dx_1.$$
 (2.3)

Here the propagator, $K(x_2, T|x_1, 0)$, which describes the transition from x_1 -ből x_2 in time T, is, says Feynman, a "sum" of the contributions (2.2) of all paths,

$$K(x_2, T|x_1, 0) = \int_{\mathcal{P}} \exp\left[\frac{i}{\hbar}S(\gamma)\right] \mathcal{D}\gamma.$$
 (2.4)

But all this intuitive, unless we say what " $\mathcal{D}\gamma$ " means here. The definition of the integration measure is indeed the main stumbling block of the whole theory, and, despite many efforts, no fully satisfactory answer is available as yet. The miracle is that the integral can, in some cases, nevertheless evaluated [6]. Below we present one possible method we illustrate on the example of a one dimensional harmonic oscillator.

Let us indeed consider an arbitrary path $\gamma(t)$ in \mathcal{P} that satisfies the given boundary conditions. Let us assume that \mathcal{P} contains a unique classical path, $\bar{\gamma}(t)$, and let us decompose γ into the sum of $\bar{\gamma}(t)$ and a

$$\gamma(t) = \bar{\gamma}(t) + \eta(t), \quad \text{ahol} \quad \eta(0) = \eta(T) = 0, \tag{2.5}$$

since the end points are kept fixed. The Lagrange function of the oscillator is $L = \frac{m}{2}(\dot{x}^2 - \omega^2 x^2)$. The action along $\gamma(t)$ is therefore

$$S(\gamma) = \int_0^T \frac{1}{2} m (\dot{\bar{\gamma}}^2 - \omega^2 \bar{\gamma}^2) dt + m \int_0^T (\dot{\bar{\gamma}} \dot{\eta} - \omega^2 \bar{\gamma} \eta) dt + \frac{1}{2} m \int_0^T (\dot{\bar{\gamma}}^2 - \omega^2 \eta^2) dt.$$
 (2.6)

For the oscillator, $\ddot{x} + \omega^2 x = 0$, so that $x(t) = A \sin \omega t + B \cos \omega t$, where the constants A and B are determined by the initial conditions,

$$B = x_1$$
 and $A = \frac{x_2 - x_1 \cos \omega T}{\sin \omega T}$,

provided $\sin \omega T \neq 0$, i.e., if $\omega T \neq N\pi$. Hence If T is not integer multiple of the half-period,

$$T \neq N \times \frac{\tau}{2}, \qquad \tau = \frac{2\pi}{\omega},$$
 (2.7)

there exists a unique classical path that starts in x_1 and arrives, after time T, to x_2 -be. The action calculated along this path is

$$S(\bar{\gamma}) = \frac{m\omega}{2\sin\omega T} \left((x_1^2 + x_2^2)\cos\omega T - 2x_1 x_2 \right). \tag{2.8}$$

Returning to (2.6), the first term is $S(\bar{\gamma})$. Next, integration by parts of the middle term yields

 $\int_0^T (\dot{\bar{\gamma}}\,\dot{\eta} - \omega^2\bar{\gamma}\,\eta)dt = -\int_0^T (\ddot{\bar{\gamma}} + \omega^2\bar{\gamma})\,\eta\,dt = 0,$

since $\bar{\gamma}$ satisfies the classical equation of motion $\ddot{x} = -\omega^2 x$. This term vanishes therefore, providing us with the propagator

$$K(x_2, T|x_1, 0) = \exp\left[\frac{i}{\hbar}S(\bar{\gamma})\right] \times F(T), \tag{2.9}$$

where the "reduced propagator", F(T), is a path integral taken over all variations,

$$F(T) = \int \exp\left\{\frac{im}{2\hbar} \int_0^T \left[(\dot{\eta}^2 - \omega^2 \eta^2) \right] dt \right\} \mathcal{D}\eta. \tag{2.10}$$

But all this is still intuitive, as we still not say what " $\mathcal{D}\eta$ " actually means. To answer this question, Feynman expands the (periodic) variation into a Fourier series,

$$\eta(t) = \sum_{k=1}^{\infty} a_k \sin[k\frac{\pi}{T}t] \quad \Rightarrow \quad \int_0^T \eta^2 dt = \frac{T}{2} \sum_k a_k^2, \qquad \int_0^T \dot{\eta}^2 dt = \frac{T}{2} \sum_k (\frac{k\pi}{T})^2 a_k^2. \tag{2.11}$$

Then he argues as follows. Any path is determined by its Fourier coefficients a_k ; instead of "integrating over all paths" let us integrate over all Fourier coefficients,

$$F(T) = \lim_{n \to \infty} \mathcal{J} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \exp\left\{ \sum_{k=1}^{n} i \frac{\lambda_k}{2\hbar} a_k^2 \right\} \times da_1 \dots da_n$$
 (2.12)

where

$$\lambda_k = m\left(\left(\frac{\pi k}{T}\right)^2 - \omega^2\right) \tag{2.13}$$

and where \mathcal{J} denotes the Jacobian of the (linear) transformation $\mathcal{P} \to \{\text{Fourier coefficients}\}$. Let us observe that \mathcal{J} is independent of the data $(\omega, m, \text{ etc})$ of the oscillator and even of \hbar .

The classical Fresnel integrals can be evaluated,

$$\int_{-\infty}^{\infty} \exp\left[i\frac{\lambda}{2}x^2\right] dx = \sqrt{\frac{2\pi}{\lambda}} e^{i\frac{\pi}{4}}.$$
 (2.14)

Then the sum in the exponent in (2.12) becomes a product,

$$F(T) = \lim_{n \to \infty} C_n \left(\prod_{k=1}^n \lambda_k \right)^{-1/2}, \tag{2.15}$$

where C_n is the product of various (divergent) factors.

The product of eigenvectors can be split into two parts.

$$\prod_{k=1}^{n} \lambda_k = \prod_{k=1}^{n} m \frac{k^2 \pi^2}{T^2} \times \prod_{k=1}^{n} \left(1 - \frac{\omega^2 T^2}{k^2 \pi^2} \right). \tag{2.16}$$

The Euler formula says now that

$$\prod_{k=1}^{\infty} \left(1 - \frac{x^2}{k^2 \pi^2} \right) = \frac{\sin x}{x},\tag{2.17}$$

so that

$$F(T) = C\sqrt{\frac{\omega T}{\sin \omega T}},$$

where C denotes the product of all ω -independent factors.

Let us now remember that for $\omega \to 0$ we get a free particle, and therefore

$$F^{free}(T) = \sqrt{\frac{m}{2\pi i\hbar T}}. (2.18)$$

This yields C, and inserting the action (2.8) the propagator becomes, at last,

$$K(x_2, T|x_1, 0) = \left(\frac{m\omega}{2\pi i\hbar \sin \omega T}\right)^{1/2} \times \exp\left\{\frac{im\omega}{2\hbar \sin \omega T} \left[\left(x_1^2 + x_2^2\right)\cos \omega T - 2x_1 x_2\right]\right\}. \tag{2.19}$$

3 The phase correction

Let us observe that if the "sin" in the denominator vanishes, the whole expression becomes meaningless. How should we then continue after such a singular point. The answer comes from the evaluation of the Fresnel integrals: Euler's formula, (2.14), is only valid for $\lambda > 0$. But

$$\lambda_k > 0 \qquad \Longleftrightarrow \qquad 0 < T < k \frac{\tau}{2}, \tag{3.1}$$

where $\tau = 2\pi/\omega$ is the period. Before reaching the first half period, $0 < T < \tau/2$, all factors under the square root are positive and Feynman's calculation is correct. After the first half period (but before a full periode), however, i. e. for $\tau/2 < T < \tau$, the first factor under the root is negative, while all the other factors remain positive.

$$\lambda_1 < 0, \quad \lambda_k > 0, \quad k \ge 2.$$

As a result, the propagator gets multiplied by

$$\frac{1}{\sqrt{-1}} = -i = e^{-i\frac{\pi}{2}}. (3.2)$$

Thus, the phase of the propagator (and therefore also of the wave function) jumps by $(-\pi/2)$. Similarly, after N but before (N+1), half-periods, i.e. for

$$N \times \frac{\tau}{2} < T < (N+1) \times \frac{\tau}{2}, \tag{3.3}$$

the first N factors in (2.15) become negative. The phase jumps, therefore, by $N \times (-\pi/2)$. The correct result is hence

$$K(x_2, T|x_1, 0) = \left(\frac{m\omega}{2\pi\hbar |\sin \omega T|}\right)^{1/2} e^{-\frac{i\pi}{4}}$$

$$\times \exp\left\{\frac{im\omega}{2\hbar |\sin \omega T|} \times \left[\left(x_1^2 + x_2^2\right)\cos \omega T - 2x_1x_2\right]\right\} \times e^{-\frac{i\pi}{2}N}.$$
(3.4)

We must admit that our argument has been somewhat sloopy: each factor $\sqrt{-1}$ could be i or -i, and choosing the second one would change every phase jump from $-\pi/2$ to $\pi/2$. Worse:

every jump can be chosen independently! Which square root of (-1) has to be chosen? During the first half-period, the question is irrelevant, since it is merely a global phase. But repeating it N times it is relevant!

The formula that generalizes (2.14) can be derived through analytic extension. For any real $\lambda \neq 0$,

$$\int_{-\infty}^{\infty} \exp\left[i\frac{\lambda}{2}x^2\right] dx = \begin{cases} \left(\frac{2\pi}{\lambda}\right)^{1/2} e^{i\frac{\pi}{4}} & \lambda > 0\\ \left(\frac{2\pi}{-\lambda}\right)^{1/2} e^{-i\frac{\pi}{4}} & \lambda < 0 \end{cases}$$
(3.5)

Euler's formula, (2.17), only holds for $x < \pi$, and otherwise it should be replaced by

$$\prod_{k=1}^{\infty} \left| 1 - \frac{x^2}{k^2 \pi^2} \right| = \frac{|\sin x|}{x}. \qquad x > 0,$$
(3.6)

This confirms the validity of our previous argument: passing through every half-peridod contributes a new negative λ , and this changes the phase by $\pi/2$.

What happens for $T = N \times \tau/2$? The propagator plainly diverges, since $\sin \omega T \to \sin N\pi = 0$ in the denominator. To derive the correct result, let us use the (semi-)group property of the time evolution,

$$U_{t+t'} = U_t \circ U_{t'} \qquad \Rightarrow \qquad U_{N\frac{\tau}{2}} = (U_{\frac{\tau}{4}})^{2N}.$$
 (3.7)

According to (2.19)

$$U_{\frac{\tau}{4}}\psi(x_2) = \left(\frac{m\omega}{2\pi\hbar}\right)^{1/2} e^{-i\pi/4} \times \int e^{-i\frac{m\omega}{\hbar}x_1x_2} \psi(x_1) dx_1, \tag{3.8}$$

which is, essentially, a Fourier-transformation. But the square of such a Fourier-transformation carries any function into itself up to a change of its argument. Hence

$$\psi_{N\frac{\tau}{2}}(x_2) = e^{-i\frac{\pi}{2}N}\psi((-1)^N x_2),\tag{3.9}$$

i.e.

$$K(x_2, T|x_1, 0) = \exp\left[-i\frac{\pi}{2}N\right] \times \delta(x_1 - (-1)^N x_2).$$
 (3.10)

Going to D dimensions, the phase will jump by $D \times \pi/2$. What happens if the oscillator is not perfectly harmonic? One has to study higher-order terms [8].

The result can be extended to several similar situations.

• For a *forced oscillator* driven by a constant external force [6, 9, 10], the previous calculation can be repeated word-by-word. The Lagrange function is

$$L = \frac{m}{2}(\dot{x}^2 - \omega^2 x^2) + fx, \tag{3.11}$$

where f = const. For $T \neq k\tau/2$ the Hamiltonian action is

$$S_{f}(\bar{\gamma}) = \frac{m\omega}{2\sin\omega T} \left((x_{1}^{2} + x_{2}^{2})\cos\omega T - 2x_{1}x_{2} \right) + 2f \frac{(1 - \cos\omega T)}{m\omega^{2}} (x_{1} + x_{2}) - f^{2} \frac{2(1 - \cos\omega T) - \omega T \sin\omega T}{m^{2}\omega^{4}}$$
(3.12)

cf. (2.8). The propagator is again (3.4) with the only change that the extra terms in (3.12) should be accounted for. In fact [10],

$$K_f(x_2, T|x_1, 0) = \exp\left[i\frac{f^2T}{2m\omega^2\hbar}\right] K_{osc}(x_2 - x^*, T|x_1 - x^*, 0)$$
(3.13)

where $x^* = f/m\omega^2$. This latter formula also holds for $T = N \times \tau/2$.

4 Derivation from the wave function

Before further investigating the various aspects and derivations of the Maslov correction from the Feynman integral viewpoint, we would like to show how it can be understood using our knowledge of the solution of the Schrödinger equation. The clue [10] is to write the propagator as

$$K(x_1, x_2; T) = \sum_{n=0}^{\infty} \exp[-i(n + \frac{1}{2})\omega T]\psi_n(x_1)\psi_n x_2,$$
(4.1)

where the ψ_n s are the normalized harmonic oscillator wavefunctions:

$$\psi_n(x) = \left(\frac{1}{2^n n!} \sqrt{\frac{m\omega}{\pi\hbar}}\right)^{1/2} \exp(-\frac{1}{2}\zeta^2) H_n(\zeta), \tag{4.2}$$

where $\zeta = \sqrt{m\omega/\hbar}x$ and the H_n denote the Hermite polynomials. Their property important for us is their behavior under space reflection, $\psi_n(-x) = (-1)^n \psi_n(x)$, implied by that of the Hermite polynomials.

(4.1) can now be evaluated. Let us assume that we are not in caustic, $t \neq N\pi/\omega$, where N is some integer. Then write $\omega t = N\pi + \omega \tau$, $0 < \tau < \pi/\omega$. N is hence the integer part of $\omega t/\pi$. Now from (4.1), we infer that

$$K = \exp(-\frac{1}{2}iN\pi) \sum_{n=0}^{\infty} \exp[-inN\pi - (i(n+\frac{1}{2})\omega\tau]\psi_n(x_1)\psi_n(x_2).$$
 (4.3)

$$K(x_1, x_2; T) = \exp(-\frac{1}{2}i\pi N) \sum_{n=0}^{\infty} \exp[-i(n + \frac{1}{2})\omega t] \psi_n(x_1) \psi_n((-1)^N x_2).$$
 (4.4)

The Hermite polynomials admit a generating function, namely

$$\mathcal{G}(\zeta_1, \zeta_2; z) = \frac{1}{\sqrt{1 - z^2}} \exp\left(\frac{2z\zeta_1\zeta_2 - z^2(\zeta_1^2 + \zeta_2^2)}{1 - z^2}\right). \tag{4.5}$$

 \mathcal{G} is analytic over the z plane with cuts going from $-\infty$ to -1 and from +1 to $+\infty$. For |z| < 1 the generating function admits the expansion

$$\mathcal{G}(\zeta_1, \zeta_2; z) = \sum_{n=0}^{\infty} \frac{z^n}{n! 2^n} H_n(\zeta_1) H_n(\zeta_2). \tag{4.6}$$

The product of the Hermite polynomials is expressed therefore, as

$$H_n(\zeta_1)H_N(\zeta_2) = 2^n \frac{d^n}{dz^n} \mathcal{G}(\zeta_1, \zeta_2; z) \Big|_{z=0}$$
 (4.7)

Then putting $z = \exp(-\alpha i)$, $0 < \alpha < \pi$, a lengthy calculation [10] yields

$$\mathcal{G} = \exp\left[\frac{1}{2}(i\alpha + \zeta_1^2 + (\zeta_2)^2)\right] \frac{e^{-i\pi/4}}{2\sin\alpha} \exp\left(\frac{i}{2\sin\alpha} \left[\zeta_1^2 + (\zeta_2)^2\cos\alpha - 2\zeta_1\zeta_2\right]\right). \tag{4.8}$$

Collecting our formulae,

$$K(x_1, x_2, T) = \sqrt{m\omega/\pi\hbar} \exp(-\frac{1}{2}\pi N[\frac{\omega t}{\pi}] \exp[-\frac{1}{2}(i\omega\tau + \zeta_1^2 + \zeta_2^2)] \mathcal{G}(\zeta, (-1)^k \zeta_2; e^{-i\omega\tau}).$$
(4.9)

Noting finally that $\sin(\omega \tau) = |\sin(\omega \tau)| = (-1)^k \sin(\omega t)$ and $\cos(\omega \tau) = (-1)^k \cos(\omega t)$ we get precisely the propagator (2.19), valid for times which are not integer multiples of half-periods.

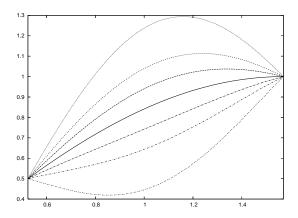


Figure 1: Those paths which join x_1 to x_2 in time T form an "infinite dimensional manifold" \mathcal{P} . The the actual motion, $\bar{\gamma}(t)$, is a critical point of the hamiltonian action, viewed as a real-valued function on \mathcal{P} .

In a caustic i.e. if $t = (\pi/\omega)N$, then, letting $\omega t \to N\pi$ in (4.4) yields [10], using the completeness of the wave functions, the formula valid in the caustic point, i.e.,

$$\lim_{\omega t \to N\pi} K(x_1, x_2; t) = \exp(-\frac{1}{2}iN\pi)\delta(x_1 - (-1)^N x_2)$$
(4.10)

cf. (3.10).

Let us mention that the Maslov correction can also be derived from a semiclassical analysis of the wave function [2].

5 How minimal is the "minimal action"?

Let us now return to classical mechanics. The classical hamiltonian action is, as explained in Section 2, a real function, $S(\gamma)$, defined on the "infinite dimensional manifold", \mathcal{P} , of all paths which join x_1 to x_2 in times T. A variation η can be viewed in turn as a "tangent vector" to \mathcal{P} in γ [11], cf. Fig. 1. δS , the first variation of the action, is the directional derivative of S in the "point" γ ,

$$\delta S_{\gamma}(\eta) = \lim_{s \to 0} \frac{S(\gamma + s\eta) - S(\gamma)}{s}.$$
 (5.1)

 δS_{γ} is, hence a one-form on \mathcal{P} . Just like in finite dimensional calculus, if $\bar{\gamma}$ is an extremal point of S, then the directional derivative must vanish in any direction,

$$\delta S_{\bar{\gamma}} \equiv \delta S_{\bar{\gamma}}(\eta) = \int \left\{ \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} \right\} \eta dt = 0.$$
 (5.2)

This yields the classical (Euler-Lagrange) equations, $\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = 0$, whose solution satisfying the boundary conditions, $\bar{\gamma}$ is, by Hamilton's Principle, is the actual motion.

But does such a solution always exist, and if it does, is it unique? The answer is, generically, affirmative. Not always, however. This point can again illustrated by the oscillator. Let us indeed assume that the elapsed time is an integer multiple of the half-period,

$$T = N \times \frac{\tau}{2}.\tag{5.3}$$

Then, independently of the initial velocity, all motions which start in x_1 arrive, at time T, to $x_2 = (-1)^N x_1$. (cf. Fig. 2.).

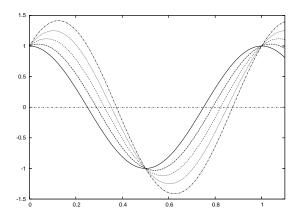


Figure 2: After a half period, all trajectories meat again, independently of the initial velocity, in the same point opposite to the one they started from.

For this particular time, (i) either i.e. when $x_2 \neq (-1)^N x_1$, there is no classical path at all with the required boundary condition, (ii) or i.e. for $x_2 = (-1)^N x_1$, there are infinitely many of them.

The solutions of the classical equation of motion behave, hence, as light rays that start from one focus of an elliptic mirror: after reflection, they are all collected into the same point, namely the other focus. (By analogy, such point are also called focal points also in variational mechanics).

Apart of this particular situation, there is a unique classical path, $\bar{\gamma}$, in \mathcal{P} . Let us assume that we are in this, generic, situation. Is the action minimal? Let us emphasized that, just like in finite dimension, (5.2) is merely a *necessary* condition for having a minimum. It is not sufficient, though. In other words, $\bar{\gamma}$ is a critical point of S, but not necessarily a minimal one.

Let us mention, at this point, that the variational calculus is in fact local, i.e., it is a differential calculus in the neighborhood of a path $\bar{\gamma}$. When calculating the first variation, we compare in fact the value of S calculated along paths which can be smoothly deformed to $\bar{\gamma}$. All such paths belong to a single path component of \mathcal{P} . If the latter has more then one path components, as it happens in the, one has to perform the variational calculus in each path-connected sector separately.

In technical terms, a "point" (i.e. a path in a manifold M) γ belongs to the path-component of $\bar{\gamma}$ if γ can joined to the "point" $\bar{\gamma}$ in \mathcal{P} if there exists a continuous "path" of "points" γ_s such that $\gamma_1 = \gamma$, $\gamma_0 = \bar{\gamma}$. But this means precisely that $\bar{\gamma}$ can be deformed into $\bar{\gamma}$ by a homotopy. The path components of \mathcal{P} are therefore labeled with the homotopy classes of the underlying space, $\pi_0(\mathcal{P}) = \pi_1(M)$.

An illustration is provided by the Aharonov-Bohm experiment [12], where M is the punctured plane $\mathbb{R}^2 \setminus \{0\}$, whose π_1 is \mathbb{Z} . The (free) action has, hence, a minimum in two homotopy classes, namely in those which pass near the solenoid (assumed infinitely thin) on either side. In all other path components, i. e. for paths which turn around the origin, the action has no critical point.

Returning to the critical points of the action, they behave just like those of a function defined over a finite (D) dimensional space : \vec{x}_0 is a critical point of a function $V(\vec{x})$ if

$$\delta V(\vec{x}_0) = 0$$
 i.e. $\partial_i V(\vec{x}_0) = 0$ $\forall i$.

Whether a given critical point \vec{x}_0 is a minimum, a maximum or a saddle point, can be determined

by looking at the matrix formed from second-order partial derivatives,

$$\delta^2 V \equiv \delta^2 V(\vec{x}_0) = \frac{\partial^2 V}{\partial x^i \partial x^j}(\vec{x}_0). \tag{5.4}$$

Is the matrix of $\delta^2 V$ definite? Being symmetric, $\delta^2 V$ will have D (real) eigenvalues,

$$\delta^2 V e_a = \lambda_a e_a, \qquad a = 1, \dots, D.$$

Then \vec{x}_0 is

$$\begin{cases} \text{a minimum if} & \lambda_a > 0 & \forall a \\ \text{a maximum if} & \lambda_a < 0 & \forall a \\ \text{saddle point if} & \lambda_a > 0 & \lambda_b < 0 & \text{for some } a, b. \end{cases}$$

$$(5.5)$$

In an analogous way, to see if a given classical path $\bar{\gamma}$ makes the action minimal one or not, we have to calculate the second variation, $\delta^2 S_{\bar{\gamma}}$. This is a quadratic form on the "tangent space" of the "variations". To be a minimum, the second variation has to be positive definite,

$$\delta^2 S_{\bar{\gamma}}(\eta, \eta) > 0 \qquad \forall \eta. \tag{5.6}$$

If, however,

$$\delta^2 S_{\bar{\gamma}}(\eta, \eta) < 0 \quad \text{but} \quad \delta^2 S_{\bar{\gamma}}(\eta', \eta') > 0$$
 (5.7)

for suitable variations η and η' , then $\bar{\gamma}$ is a saddle point.

A variation η such that

$$\delta^2 S_{\gamma}(\eta, \eta) < 0 \tag{5.8}$$

is a negative mode, and if

$$\frac{1}{2}\delta^2 S(\eta, \eta') = 0 \qquad \forall \eta', \tag{5.9}$$

it is a zero-mode. In the direction of a zero mode the function is, in the first approximation, invariant. In this case, the nature of the critical point depends on the higher variations.

Let us examine the second variation in some detail.

$$\frac{1}{2}\delta^2 S \equiv \frac{1}{2}\delta^2 S_{\bar{\gamma}}(\eta,\eta) = \frac{1}{2}\int_0^T \left\{ \frac{\partial^2 L}{\partial x^2} \eta^2 + 2 \frac{\partial^2 L}{\partial x \partial \dot{x}} \eta \dot{\eta} + \frac{\partial^2 L}{\partial x^2} \eta^2 + \frac{\partial^2 L}{\partial \dot{x}^2} \dot{\eta}^2 \right\} dt,$$

where the integration is along the (assumed unique) classical path $\bar{\gamma}$. After partial integration,

$$\frac{1}{2}\delta^2 S = \int_0^T (\eta, \Lambda \eta) dt,$$

$$\Lambda = -\frac{d}{dt} \left(\frac{\partial^2 L}{\partial \dot{x}^2} \frac{d}{dt} + \frac{\partial^2 L}{\partial x \partial \dot{x}} \right) + \left(\frac{\partial^2 L}{\partial x \partial \dot{x}} \frac{d}{dt} + \frac{\partial^2 L}{\partial x^2} \right). \tag{5.10}$$

where Λ is the operator of the second variation. $\frac{1}{2}\delta^2 S$ is hence positive definite if all eigenvalues of the quadratic form Λ are positive, $\lambda > 0$, for all

$$\Lambda \eta = \lambda \eta, \qquad \eta(0) = \eta(T) = 0. \tag{5.11}$$

• In the simplest possible case of a one-dimensional massive particle in a potential,

$$L = \frac{m\dot{x}^2}{2} - V(x) \qquad \Rightarrow \qquad \Lambda = -m\left(\frac{d^2}{dt^2} + \frac{d^2V}{dx^2}\right). \tag{5.12}$$

For the 1D oscillator, e.g.,

$$\Lambda_{osc} = -m \left(\frac{d^2}{dt^2} + \omega^2 \right). \tag{5.13}$$

Hence, taking into account the boundary conditions,

$$m(\ddot{\eta} + \omega^2 \eta) = -\lambda \eta \quad \Rightarrow \quad \eta(t) = \sin[(\omega^2 + \frac{\lambda}{m})^{1/2} t].$$

But $T\sqrt{\omega^2 + \frac{\lambda}{m}} = k\pi$, $k = 0, \pm 1, \dots$ due to periodicity, so that

$$\lambda_k = m\left(\left(\frac{k\pi}{T}\right)^2 - \omega^2\right) \quad \text{and} \quad \eta_k(t) = \sin(\frac{k\pi}{T}t),$$
 (5.14)

cf. (2.11) and (2.13). The integer N > 0 in (3.3) $N\tau/2 < T < (N+1)\tau/2$, counts hence the negative eigenvalues. Thus, the oscillator trajectory is a minimum of the action during the first half period. For time beyond $N\tau/2$, it becomes a saddle point with N negative modes!

What happens in the focal points? For $T=N\times \tau/2$, as seen above, either we have no classical path at all or we have infinitely many between our fixed endpoints. Let $x_2=(-1)^Nx_1$ and let us consider some $\bar{\gamma}$. Then all the other classical paths can be viewed as a variation of $\bar{\gamma}$, labeled by a parameter $s, \gamma = \gamma_s$. The action calculated for any such classical path $\gamma = \gamma_s$ is the same. Varying s we get, hence, a "curve" in \mathcal{P} inscribed onto the "level surface" S= const. The derivative w.r.t. s of $S(\gamma_s)$ is therefore zero, so that

$$\eta(t) = \frac{d}{ds} \gamma_s(t) \bigg|_{s=0}$$
(5.15)

is a null mode at $\bar{\gamma}$.

Minimum or not: what are the physical consequences? At the purely classical level, nothing at all: (5.2) yields the correct equations of motion in all cases. In Quantum Mechanics, however, the consequence is precisely the Maslov phase jump, as we explain it below.

Let us first consider another example.

• A phase jump similar to the one found for the oscillator is observed for a charged particle moving perpendicularly to the induction lines of a constant magnetic field [6]. Classically, the particle rotates uniformly with Larmor frequency eB/m i.e. with period $\tau = 2\pi m/eB$. The Lagrange function is

$$L_B = \frac{m}{2} \left(\dot{x}^2 + \dot{y}^2 + 2\omega(x\dot{y} - y\dot{x}) \right), \qquad \omega = \frac{eB}{2m}.$$
 (5.16)

If T is not an integer multiple of a (full) period, $T \neq N \times \tau = \frac{2N\pi m}{eB}$, then there is a unique classical trajectory that links any two points (x_1, y_1) and (x_2, y_2) in time T. The action calculated for it is

$$S_B = \frac{m\omega}{2} \left[\left[(x_2 - x_1)^2 + (y_2 - y_1)^2 \right] \cot(\omega T) + 2(x_1 y_2 - x_2 y_1) \right]. \tag{5.17}$$

The matrix of the second variation is now

$$\Lambda_B = m \begin{pmatrix} \frac{d^2}{dt^2} & -2\omega \frac{d}{dt} \\ 2\omega \frac{d}{dt} & \frac{d^2}{dt^2} \end{pmatrix}.$$
 (5.18)

Our task is to solve the eigenvalue problem

$$\begin{cases}
 m\ddot{\eta}_x - 2m\omega\dot{\eta}_y = -\lambda\eta_x \\
 m\ddot{\eta}_y + 2m\omega\dot{\eta}_x = -\lambda\eta_y
\end{cases}, \quad \eta_x(0) = \eta_y(0) = 0 = \eta_x(T) = \eta_y(T), \quad (5.19)$$

The solutions are readily derived by separating the equations (5.19) by applying a timedependent rotation,

$$\begin{cases} \eta_x = \cos \omega t \, \xi + \sin \omega t \, \zeta \\ \eta_y = -\sin \omega t \, \xi + \cos \omega t \, \zeta \end{cases} \Rightarrow \begin{cases} m\ddot{\xi} + (m\omega^2 + \lambda)\xi = 0 \\ m\ddot{\zeta} + (m\omega^2 + \lambda)\zeta = 0 \end{cases}$$

By periodicity, $\xi, \zeta \propto \sin(\sqrt{\omega^2 + \lambda} t) = \sin\frac{k\pi}{T}t$, where k is some integer. The eigenvalues are, therefore, doubly degenerate, and are identical to those in the oscillator problem:

$$\lambda_k = m\left(\left(\frac{k\pi}{T}\right)^2 - \omega^2\right), \qquad k = 0, 1, \dots$$
 (5.20)

The reduced propagator is hence identical to that of a planar oscillator whose frequency is half of the Larmor value, $\omega = eB/2m$.

If
$$N\tau < T < (N+1)\tau$$
, then

$$K(x_{2}, y_{2}, T | x_{1}, y_{1}0) = \frac{m\omega}{2\pi i\hbar |\sin \omega T|} \times \exp\left\{\frac{im\omega}{2\hbar} \left[\left[(x_{2} - x_{1})^{2} + (y_{2} - y_{1})^{2} \right] \cot(\omega T) + 2(x_{1}y_{2} - x_{2}y_{1}) \right] \right\} (-1)^{N}, \quad \omega = \frac{eB}{2m}.$$
(5.21)

cf. [13, 10]. After N full periods i.e. at $T = \frac{2N\pi m}{eB}$ [which corresponds to N half-oscillator-periods], all classical motions meet again in the point they started from. According to our previous results, taking into account the dimension of the problem, the propagator is again a Dirac-delta with a sign change:

$$K(x_2, y_2, \tau | x_1, y_1, 0) = (-1)^N \delta(x_1 - x_2, y_1 - y_2).$$
(5.22)

It is worth mentioning that this calculation should turn out to be useful to explain the Sagnac effect [14]. This experiment, originally proposed and performed with light and later been repeated with massive particles, amounts to perform a two-slit type interference experiment, when the whole apparatus is fixed on a turntable. The clue is that the inertial force due to rotation behaves exactly as a fictious magnetic field, with twice the mass, 2m, replacing the electric charge.

6 The Semiclassical approximation

Returning to the general case, let us assume that \mathcal{P} has a unique classical path, $\bar{\gamma}$, and let us develop the action to second order [15]:

$$S(\gamma) = S(\bar{\gamma}) + \delta_{\bar{\gamma}}(\eta) + \frac{1}{2}\delta^2 S_{\bar{\gamma}}(\eta, \eta) + \dots$$
(6.1)

where the "dots, ...", denote all higher-order variations.

Taking the semiclassical approximation amounts of dropping all these terms. According to Hamilton's Principle $\delta S_{\bar{\gamma}} = 0$. The semiclassical propagator is, hence,

$$K(\vec{x}_2, T | \vec{x}_1, 0) = \exp\left[\frac{i}{\hbar}S(\bar{\gamma})\right] \times F(T), \qquad F(T) = \int \exp\left\{\frac{i}{2\hbar}\delta_{\bar{\gamma}}^2 S(\eta, \eta)\right\} \mathcal{D}\eta. \tag{6.2}$$

cf. (2.9)-(2.10).

The reduced propagator can be determined diagonalizing the second variation. Let us assume, for simplicity, that the system is 1-dimensional. The quadratic form Λ in the eigenvalue equation (5.11) is a self-adjoint Sturm-Liouville operator on the space of all "tangent vectors" η . The eigenvalues, λ_n , are therefore all real and form a complete orthonormal system w.r.t. the usual scalar product, $(\eta_n, \eta_m) = T^{-1} \int \eta_n \eta_m dt = \delta_{nm}$. Expanding the variation η as $\eta = \sum_n a_n \eta_n$ yields $(\eta, \Lambda \eta) = \sum_k a_k^2 \lambda_k$. The reduced propagator is, therefore, once again (2.12), with the λ_k denoting the eigenvectors of Λ in (5.11). Hence

$$F(T) = C\sqrt{\prod_{k} \frac{2i\pi\hbar}{\lambda_{k}}} \ . \tag{6.3}$$

As explained before, the Jacobian is independent of the dynamics, so that (6.3) holds also for the free factor,

$$F(T)^{free} = C\sqrt{\prod_{k} \frac{2i\pi\hbar}{\lambda_{k}^{free}}},$$
(6.4)

where $F(T)^{free}$ and λ_k^{free} are the free propagator and the eigenvalues, respectively. Dividing (6.3) by (6.4) C drops out,

$$F(T) = F(T)^{free} \times \sqrt{\prod_{k} \lambda_{k}^{free} / \prod_{k} \lambda_{k}}$$
.

The semiclassical propagator is, therefore,

$$K = e^{iS(\bar{\gamma})} \left(\frac{m}{2\pi i\hbar T}\right)^{1/2} \times \sqrt{\frac{\prod_k \lambda_k^{free}}{\prod_k \lambda_k}} , \qquad (6.5)$$

since, in D dimensions, $F(T)^{free} = \left(\frac{m}{2\pi i \hbar T}\right)^{D/2}$, cf. (2.18).

• For a 1D oscillator, we recover the previous result : by (2.16), the quotient of the products of the eignevalues under the root is exactly the infinite product we determined using in the Euler formula, since $\lambda_k^{free} = m\pi^2 k^2/T^2$.

The Van Vleck matrix.

Our result can also be presented using the $Van\ Vleck\ matrix\ [15]$. Let us indeed remember that, still assuming the uniqueness of the classical path, $\bar{\gamma}$, between the to given points \vec{x}_1 -et \vec{x}_2 in time T, the action can be viewed as function of the end points.

$$S(\vec{x}_1, \vec{x}_2) = S(\bar{\gamma}) \tag{6.6}$$

is in fact Hamilton's Principal Function. The determinant of the $D \times D$ matrix

$$\left[\frac{\partial^2 S}{\partial \vec{x}_1 \partial \vec{x}_2}\right] = \left[\frac{\partial^2 S}{\partial x_1^i \partial x_2^j}\right] \tag{6.7}$$

is called the Van Vleck determinant. Then

Theorem ([15]): In D dimensions, the semiclassical propagator is

$$K(\vec{x}_2, T | \vec{x}_1, 0) = \left(\frac{1}{2\pi i\hbar}\right)^{1/2} \left| \det \frac{\partial^2 S}{\partial \vec{x}_1 \partial \vec{x}_2} \right|^{D/2} \exp\left[-iDN\frac{\pi}{2}\right] \times \exp\left[\frac{i}{\hbar}S(\bar{\gamma})\right]. \tag{6.8}$$

where $\vec{x}_1 = (x_1^i)$ and $\vec{x}_2 = (x_2^j)$ are the initial and final point, respectively.

 \bullet For the 1D oscillator

$$\frac{\partial^2 S}{\partial x_1 \partial x_2} = -\frac{m\omega}{\sin \omega t},\tag{6.9}$$

so that (6.8) yields, once again, the previous oscillator propagator formula.

• Let us consider a charged particle in a constant magnetic field, and let us assume that $T \neq \tau$. Hamilton's Principal function is now (5.17), and the absolute value of the Van Vleck determinant reads

$$|\det \frac{\partial^2 S}{\partial \vec{x}_1 \partial \vec{x}_2}| = m^2 \omega^2 \times \det \begin{pmatrix} -\cot \omega T & 1\\ -1 & -\cot \omega T \end{pmatrix} = \frac{m^2 \omega^2}{\sin^2 \omega T}.$$
 (6.10)

From (6.8) we get (5.21) once again.

7 Morse theory [16]

The methods of variational calculus allow us to further deepen our understanding. Let us chose a classical motion $\bar{\gamma}$ [in *D*-dimensions], and let us focus our attention to the second variation. As seen in (5.10)

$$\frac{1}{2}\delta^2 S(\eta, \eta) = \int_0^T (\eta, \Lambda \eta) dt,$$

where Λ is the matrix of the second variation. A Jacobi field along $\bar{\gamma}$ is a D-dimensional vector $\vec{\xi}(t) = (\xi_i(t))$ such that

$$\Lambda \vec{\xi} = 0$$
 i.e. $\Lambda_{ij} \xi_j = 0 \quad \forall i.$ (7.1)

This second-order differential equation in 2D dimensions has 2D independent solutions. Generally, none of them belongs to the tangent space of \mathcal{P} at $\bar{\gamma}$, since $\bar{\xi}(0)$ and/or $\bar{\xi}(T)$ may not vanish. Two points $p = \bar{\gamma}(t_1)$ and $q = \bar{\gamma}(t_1)$ of a curve $\bar{\gamma}$, $0 \le t_1 < t_2 \le T$ are called conjugate point, if there exists a Jacobi field such that

$$\vec{\xi}(t_1) = \vec{\xi}(t_2) = 0, \tag{7.2}$$

i.e., if it belongs to the tangent space. The number of such independent fields is called the multiplicity of the conjugate points p and q.

The null space of the second variation is the set of all tangent vectors η for which $\frac{1}{2}\delta^2 S(\eta, \eta') = 0$ for all η' . The dimension, ν , of the null space is thus the number of independent Jacobi fields that also belong to the tangent space. The null-space is, hence, non-trivial, if the starting and the ending points of $\bar{\gamma}$, $p = x_1 = \bar{\gamma}(0)$ and $q = x_2 = \bar{\gamma}(T)$, are conjugate. ν is finite and it is readily seen that $\nu < D$ [16].

It is easy to see that 2D independent Jacobi fields can be constructed using classical paths. Let us indeed consider a one-parameter family of classical paths [i.e. solutions of the Euler-Lagrange equations] $\beta(u)$, $0 \le u \le 1$, such that $\beta(0) = \bar{\gamma}$. These paths are not required to have the same end points as $\bar{\gamma}$. Then it is easy to see that

$$\vec{\xi} = \frac{\partial \beta}{\partial u} \Big|_{u=0} \tag{7.3}$$

is a Jacobi field along $\bar{\gamma}$. The general solution of the Euler-Lagrange equations depends on 2D parameters. Deriving w.r.t. these parameters yields 2D independent Jacobi fields.

Let us now define the *Morse index* of $\bar{\gamma}$ as the maximal dimension, μ , of those subspaces of the tangent space upon which the restriction of the second variation is negative definite.

$$\delta^2 S_{\gamma}(\eta, \eta) < 0. \tag{7.4}$$

Then we have:

Theorem (Morse) [16]: Morse index is the number of those points of $\bar{\gamma}$, each counted with its multiplicity, which are conjugate to $x_1 = \bar{\gamma}(0)$.

The essence of the proof is to show that the number of negative eigenvalues is precisely the Morse index $\mu = \mu(T)$. This is proved by showing that the Morse index, $\mu(t)$, calculated on the segment from t = 0 to t of the curve is a monotonic function of t which, for sufficiently small t, is continuous from the left, $\mu(t - \epsilon) = \mu(t)$ for $\epsilon > 0$ sufficiently small. From the right, instead,

$$\mu(t+\epsilon) = \mu(t) + \nu,\tag{7.5}$$

where ν is, as before, the dimension of the null space of the second variation. The $\mu(t)$ as a function of time is hence, at first, constant, and then, upon crossing a conjugate point, it jumps by ν .

The Morse index is finite, so that our curve contains a finite number of points conjugate to the starting point.

Let us again illustrate the general theory on example.

• For the 1D oscillator and for Λ in (5.13), the solutions of the Jacobi equation (7.1) are $a \sin \omega t + b \cos \omega t$, which vanish at t = 0 and at t = T if b = 0 and $T\omega = N\pi$. A point q of the classical trajectory is conjugate to $x_1 = \bar{\gamma}(0)$ if

$$q = q_k = \bar{\gamma}(k\frac{\tau}{2}), \qquad k = 1, \dots$$

Consistently with our general theorem, the null space of $\delta^2 S$ can be also obtained by considering the classical motion which starts from x_1 with velocity v,

$$\frac{v}{\omega}\sin\omega t + x_1\cos\omega t \quad \Rightarrow \quad \xi(t) = \frac{\partial x(t)}{\partial v} = \frac{1}{\omega}\sin\omega t. \tag{7.6}$$

For $t = N\tau/2$, we get manifestly a null-mode of the second variation.

For a D-dimensional oscillator, each conjugate point contributes D null-modes. When T crosses $2N/\tau$ the dimension of the null space, i.e., the Morse index jumps by D.

Let us mention that, on compact manifolds, the Morse index is related to the topology of the infinite dimensional manifold of paths [16].

8 Conclusion

Is the phase observable? The naive answer to this question is negative: when the wave function is multiplied by any phase factor, the absolute value of the amplitude, and hence its square, the probability, are unchanged. This is only true for one wave function, though. If the system is decomposed into parts and the partial waves are multiplied by different phase factors, recombination will yield an interference.

Similar phase jumps are found in molecular [17], nuclear [18], and heavy ion [19] collisions.

After the original ideas reviewed here have been put forward, several developments took place. See, e.g. [20, 21]. Related questions are also discussed in some textbooks [22].

It is worth calling the attention to that all path integrals studied in this paper can be calculated [23] by transforming the system into the free form using a "non-relativistic conformal transformation" [24, 25]. For the harmonic oscillator, for example, every half-period can be conformally related to a full free motion. Then the Maslov correction can be recovered by fitting together the wave function at half-periods [25]. This latter paper also provides a complete catalog of all possibilities.

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